

# Separability of qubit-qudit quantum states with strong positive partial transposes

Kil-Chan Ha<sup>1</sup>

<sup>1</sup>*Faculty of Mathematics and Statistics, Sejong University, Seoul 143-747, Korea*

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We show that all  $2 \otimes 4$  states with strong positive partial transposes (SPPT) are separable. We also construct a family of  $2 \otimes 5$  entangled SPPT states, so the conjecture on the separability of SPPT states are completely settled. In addition, we clarify the relation between the set of all  $2 \otimes d$  separable states and the set of all  $2 \otimes d$  SPPT states for the case of  $d = 3, 4$ .

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The notion of quantum entanglement plays a key role in the current study of quantum information and quantum computation theory. One of the central problems in the theory of quantum entanglement is to check whether a given density matrix representing a quantum state of composite system is separable or entangled. Let us recall that a state  $\rho$  acting on the Hilbert space  $\mathcal{H}_A \otimes \mathcal{H}_B$  is called separable if it is a convex combination of product states, that is,  $\rho = \sum_k p_k \rho_k \otimes \tilde{\rho}_k$ , where  $\rho_k$  and  $\tilde{\rho}_k$  are states acting on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively [1].

There are two main criteria for separability: One criterion was given by Horodecki *et al.* [2] using positive linear maps between matrix algebras, and this was formulated as the notion of entanglement witnesses [3]. Another criterion, the so called partial positive transpose (PPT) criterion [4] tells us that if a state  $\rho$  is separable then its partial transposition  $\rho^{T_A} = (T \otimes \mathbb{1})\rho$  is positive. It was shown by Horodecki *et al.* [5] that the PPT criterion is also a sufficient condition for separability in the system  $2 \otimes 2$  and  $2 \otimes 3$ . For higher dimensions, this is not the case by a work of Woronowicz [6] who gave an example of a  $2 \otimes 4$  PPT entangled state. Such examples were also given in Refs. [7, 8] for the  $3 \otimes 3$  cases, in the early eighties. See also Ref. [9] for the  $2 \otimes 4$  case. Therefore, it is important to understand which PPT states are separable and which are entangled.

In this context, a subclass of PPT states, whose PPT properties are ensured by the canonical construction using Cholesky decomposition, were considered in Ref. [10]. Such states are called strong positive partial transpose (SPPT) states. Based on several examples of SPPT states, it was conjectured in [10] that all SPPT states are separable. Unfortunately this is not true in general  $m \otimes n$  SPPT states with  $m, n > 2$  since there exist entangled  $3 \otimes 3$  states which are SPPT [11]. However, this conjecture is still open for  $2 \otimes d$  SPPT states. We note that  $2 \otimes d$  systems are particularly useful since it allows us to determine a number of separability properties in the multiqubit case [12], and were intensively analyzed in [13]. See also Ref. [14] in which it was proved that SPPT notion can be used for witnessing quantum discord in  $2 \otimes d$  systems.

In this Brief Report, we show that all  $2 \otimes 4$  SPPT states are separable, but the conjecture proposed in [10] does not hold true for general  $2 \otimes d$  systems with  $d \geq 5$ . This

result displays the difference between the  $2 \otimes d$  case and the general  $m \otimes n$  one ( $m, n > 2$ ), even difference between  $2 \otimes 4$  and  $2 \otimes 5$  cases. We also clarify the relation between the notion of separability and the SPPT property.

We begin with the definition of a  $2 \otimes d$  SPPT state. Consider the following class of upper triangular block matrices  $\mathbf{X}$  and  $\mathbf{Y}$ :

$$\mathbf{X} = \begin{pmatrix} X_1 & SX_1 \\ 0 & X_2 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} X_1 & S^\dagger X \\ 0 & X_2 \end{pmatrix}, \quad (1)$$

where  $X_k$  and  $S$  are arbitrary  $d \times d$  matrices. Then we say that a state

$$\rho = \mathbf{X}^\dagger \mathbf{X} = \begin{pmatrix} X_1^\dagger X_1 & X_1^\dagger S X_1 \\ X_1^\dagger S^\dagger X_1 & X_1^\dagger S^\dagger S X_1 + X_2^\dagger X_2 \end{pmatrix} \quad (2)$$

with  $\mathbf{X}$  defined in (1) has SPPT if  $\rho^{T_A} = \mathbf{Y}^\dagger \mathbf{Y}$  with  $\mathbf{Y}$  defined in (1). It is clear that  $\rho = \mathbf{X}^\dagger \mathbf{X}$  has SPPT if and only if

$$X_1^\dagger S^\dagger S X_1 = X_1^\dagger S S^\dagger X_1. \quad (3)$$

We also note that  $\rho$  has SPPT if and only if  $(\mathbb{1} \otimes V)^\dagger \rho (\mathbb{1} \otimes V)$  has SPPT with nonsingular  $d \times d$  matrix.

First, we consider  $2 \otimes d$  SPPT state  $\rho$  with  $r(X_1^\dagger X_1) = d$ , where  $r(X)$  denotes the rank of  $X$ . In this case,  $X_1$  is nonsingular, and so  $S$  is normal matrix by the condition (3). Thus we have the spectral decomposition for  $S$

$$S = \sum_{i=1}^d \lambda_i P_i,$$

where  $P_i$ 's are rank one projections with  $\sum_{i=1}^d P_i = \mathbb{1}$ . Then we can write

$$\begin{pmatrix} \mathbb{1} & S \\ S^\dagger & S^\dagger S \end{pmatrix} = \sum_{i=1}^d \sigma_i \otimes P_i \quad \text{with } \sigma_i = \begin{pmatrix} 1 & \lambda_i \\ \lambda_i^* & |\lambda_i|^2 \end{pmatrix}$$

and we see that

$$\begin{pmatrix} X_1^\dagger & 0 \\ 0 & X_1^\dagger \end{pmatrix} \begin{pmatrix} \mathbb{1} & S \\ S^\dagger & S^\dagger S \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix} = \sum_{i=1}^d (\sigma_i \otimes X_1^\dagger P_i X_1)$$

is an unnormalized separable state. Therefore, we can conclude that

$$\rho = \begin{pmatrix} X_1^\dagger & 0 \\ 0 & X_1^\dagger \end{pmatrix} \begin{pmatrix} \mathbb{1} & S \\ S^\dagger & S^\dagger S \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X_2^\dagger X_2 \end{pmatrix}$$

is separable since the second matrix in the righthand side is  $|1\rangle\langle 1| \otimes X_2^\dagger X_2$ , where  $|1\rangle = (0, 1)^\dagger$ . Consequently we have the following.

**Proposition 1** *Let  $\rho$  be a  $2 \otimes d$  SPPT state of the form (2) with  $r(X_1^\dagger X_1) = d$ . Then  $\rho$  is separable.*

From the above Proposition, we obtain a sufficient condition for separability of  $2 \otimes d$  PPT states  $\rho$  with  $r(\langle 0|\rho|0\rangle) = d$ . To see this, we observe the condition when such a PPT state is SPPT. Let  $\rho$  be a  $2 \otimes d$  PPT state of the form

$$\rho = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \quad \text{with } r(A) = d. \quad (4)$$

From the PPT property of  $\rho$ , we see that both  $C - B^\dagger A^{-1} B$  and  $C - B A^{-1} B^\dagger$  are positive semi-definite matrices (see the Theorem 1.3.3 in Ref. [15]). So we can find  $X_2$  satisfying the following condition

$$C - B^\dagger A^{-1} B = X_2^\dagger X_2$$

since  $A$  is a invertible positive definite matrix. Thus we have

$$\rho = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} = \begin{pmatrix} X_1^\dagger X_1 & X_1^\dagger \tilde{B} X_1 \\ X_1^\dagger \tilde{B}^\dagger X_1 & X_1^\dagger \tilde{B}^\dagger \tilde{B} X_1 + X_2^\dagger X_2 \end{pmatrix},$$

where  $X_1 = A^{1/2} = (A^\dagger)^{1/2}$ ,  $\tilde{B} = (A^\dagger)^{-1/2} B A^{-1/2}$ . Consequently, we have the following result from the condition (3).

**Corollary 2** *Let  $\rho$  be a  $2 \otimes d$  PPT state of the form (4). Then  $\rho$  is SPPT if and only if the condition  $B^\dagger A^{-1} B = B A^{-1} B^\dagger$  is satisfied. In this case, the PPT state  $\rho$  is separable.*

Now, we consider a  $2 \otimes d$  SPPT state  $\rho$  with  $r(X_1^\dagger X_1) = k < d$ . Then we may write  $X_1 = U \Sigma V^*$  for some  $d \times d$  unitary matrices  $U, V$  and a diagonal matrix  $\Sigma$  of rank  $k$  with diagonal entries  $\sigma_{11} \geq \sigma_{22} \geq \dots \geq \sigma_{kk} \geq 0$  by the singular value decomposition. Thus  $\rho$  in (2) can be written by

$$\begin{aligned} \rho &= \begin{pmatrix} X_1^\dagger X_1 & X_1^\dagger S X_1 \\ X_1^\dagger S^\dagger X_1 & X_1^\dagger S^\dagger S X_1 + X_2^\dagger X_2 \end{pmatrix} \\ &= \begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \Sigma^2 & \Sigma \tilde{S} \Sigma \\ \Sigma \tilde{S}^\dagger \Sigma & \Sigma \tilde{S}^\dagger \tilde{S} \Sigma \end{pmatrix} \begin{pmatrix} V^\dagger & 0 \\ 0 & V^\dagger \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X_2^\dagger X_2 \end{pmatrix}, \end{aligned} \quad (5)$$

where  $\tilde{S} = U^\dagger S U$ . Now, we write  $\Sigma$  and  $\tilde{S}$  as block matrices

$$\Sigma = \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{pmatrix},$$

where  $D_k$  and  $\tilde{S}_{11}$  are  $k \times k$  matrices. Then we have

$$\begin{aligned} \tilde{\rho} &:= \begin{pmatrix} \Sigma^2 & \Sigma \tilde{S} \Sigma \\ \Sigma \tilde{S}^\dagger \Sigma & \Sigma \tilde{S}^\dagger \tilde{S} \Sigma \end{pmatrix} \\ &= \begin{pmatrix} D_k^2 & 0 & D_k \tilde{S}_{11} D_k & 0 \\ 0 & 0 & 0 & 0 \\ D_k \tilde{S}_{11}^\dagger D_k & 0 & D_k (\tilde{S}_{11}^\dagger \tilde{S}_{11} + \tilde{S}_{21}^\dagger \tilde{S}_{21}) D_k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

with  $D_k (\tilde{S}_{11}^\dagger \tilde{S}_{11} + \tilde{S}_{21}^\dagger \tilde{S}_{21}) D_k = D_k (\tilde{S}_{11} \tilde{S}_{11}^\dagger + \tilde{S}_{12} \tilde{S}_{12}^\dagger) D_k$  by the condition (3). It is easy to see that  $\tilde{\rho}$  is unnormalized separable state if and only if the following reduced unnormalized  $2 \otimes k$  state

$$\begin{pmatrix} D_k^2 & D_k \tilde{S}_{11} D_k \\ D_k \tilde{S}_{11}^\dagger D_k & D_k (\tilde{S}_{11}^\dagger \tilde{S}_{11} + \tilde{S}_{21}^\dagger \tilde{S}_{21}) D_k \end{pmatrix} \quad (6)$$

is separable. We note that the above  $2 \otimes k$  state is a PPT state, although it may not be SPPT. Therefore, if  $k \leq 3$  then we see that  $\tilde{\rho}$  is separable. In this case, we can conclude that  $\rho$  is separable in (5). Consequently, we have the following.

**Proposition 3** *Let  $\rho$  be a  $2 \otimes d$  SPPT state of the form (2) with  $r(X_1^\dagger X_1) \leq 3$ . Then  $\rho$  is separable.*

To answer the conjecture asked in [10], we will show the following.

**Theorem 4** *All  $2 \otimes d$  states with strong positive partial transpose are separable if and only if  $d \leq 4$ .*

*Proof.* By combining the Proposition 1 and 2, we see that all  $2 \otimes 4$  SPPT states are separable. To complete the proof, we construct a  $2 \otimes 5$  SPPT state which is not separable. Define  $5 \times 5$  matrices  $X_1$  and  $S$  by

$$X_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 \\ \beta_2 & 0 & 0 & \beta_1 & 0 \end{pmatrix}$$

where  $\beta_1 = [(1-b)/2b]^{1/2}$  and  $\beta_2 = [(1+b)/2b]^{1/2}$  with  $0 < b < 1$ . We also put  $X_2$  by  $5 \times 5$  zero matrix. Then we define  $2 \otimes 5$  SPPT state  $\varrho_0$  by Eqs. (1) and (2):

$$\varrho_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \gamma_1 & 0 & 0 & \gamma_2 & \cdot \\ 1 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot & 0 & 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \gamma_2 & 0 & 0 & \gamma_1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where  $\gamma_1 = (b+1)/(2b)$ ,  $\gamma_2 = \sqrt{b^2-1}/(2b)$ , and  $\text{dot}(\cdot)$  denotes zero. We note that the corresponding reduced  $2 \otimes 4$  state as in (6) is the PPT entangled state given by Horodecki [9]. Therefore, we conclude that  $\varrho_0$  is an entangled state with strong positive partial tranpose. This completes the proof.  $\square$

Lastly, for  $d = 3, 4$ , we show that the set of all  $2 \otimes d$  SPPT states is proper subset of the set of all separable  $2 \otimes d$  states. To see this, we consider a  $2 \otimes 3$  state  $\varrho_1 =$

$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^\dagger & \rho_{22} \end{pmatrix}$  defined by

$$\rho_{11} = \begin{pmatrix} 3 & \cdot & \cdot \\ \cdot & 4 & 2 \\ \cdot & 2 & 3 \end{pmatrix}, \rho_{12} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ 1 & -1 & \cdot \end{pmatrix}, \rho_{22} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 6 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Then, we can easily check that four  $3 \times 3$  matrices  $\rho_{11}$ ,  $\rho_{22}$ ,  $\rho_{22} - \rho_{12}^\dagger \rho_{11}^{-1} \rho_{12}$  and  $\rho_{22} - \rho_{12} \rho_{11}^{-1} \rho_{12}^\dagger$  are all positive definite matrices. Therefore, we see that  $\varrho_1$  is  $2 \otimes 3$  PPT state, and so  $\varrho_1$  is separable. On the other hand, we see that

$$\rho_{12}^\dagger \rho_{11}^{-1} \rho_{12} - \rho_{12} \rho_{11}^{-1} \rho_{12}^\dagger = \frac{1}{12} \begin{pmatrix} 6 & -6 & -3 \\ -6 & 0 & 0 \\ -3 & 0 & -4 \end{pmatrix}.$$

Therefore,  $\varrho_1$  is not SPPT by the Corollary 2. Now, we define  $2 \otimes 4$  state  $\rho_2$  using the above  $2 \otimes 3$  state  $\rho_1$  as follows:

$$\rho_2 = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} & 0 \\ 0 & 1 & 0 & 0 \\ \rho_{12}^\dagger & 0 & \rho_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is obvious that  $\varrho_2$  is separable. We can also show that  $\varrho_2$  is not SPPT by the Corollary 2. This completes the proof of claim.

In conclusion, we showed that all  $2 \otimes 4$  SPPT states are separable. We also constructed a family of  $2 \otimes 5$  SPPT entangled states using Horodecki's  $2 \otimes 4$  PPT entangled states. So the conjecture [10] on the separability of SPPT states is completely settled. We also clarify the relation between the set of all  $2 \otimes d$  separable states and the set of all  $2 \otimes d$  SPPT states for the case of  $d = 3, 4$ .

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